SOME PROBLEMS OF THE MATHEMATICAL THEORY OF DEFORMATION AND FRACTURE OF HARD ROCKS

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Hard rocks exhibit brittleness under the influence of high intensity loading; fracture occurs by the formation of a large number of cracks. In addition, such rocks have elastic-plastic properties. In this paper, the simplest model is constructed for mathematical description of the deformation and motion of this kind of material. In the framework of this model, the problem of the effect of the explosion of a concentrated explosive charge in a brittle rock is examined. In Section 1 general considerations are presented for the construction of a mathematical model of the medium. The remaining sections contain the formulation and solution of the problem of the effect of the explosion of a concentrated charge in a hard rock. The analysis of the basic problem for various typical cases is given in Sections 2 and 4. The energy equation is examined in Section 3.

1. In materials which do not have pores (including the majority of hard rocks), brittle fracture can occur by the formation of tension cracks and shear cracks. Moreover, in porous brittle rocks (e.g. coquina), failure (fracture) can also occur under hydrostatic compression because of failure of the brittle porous skeleton.

The critical state which precedes actual failure by the formation of cracks (or by crushing in the case of porous rock) can be described in the form of some invariant expression which relates the components of the stress tensor — the failure criterion. In the general case for an isotropic material this relation is written in the form

$$\Phi\,(I_1,\ I_2,\ I_3) \leqslant 0 \tag{1.1}$$

where I_1, I_2, and I_3 are independent invariants of the stress tensor (these may be, for instance, the principal stresses $\sigma_1, \sigma_2, \sigma_3$). If the state of stress of an element is such that the inequality holds in the condition (1.1), then we shall consider that the element is in a sound state. When the equality is attained in (1.1) the limit of strength is reached in the element and it fails; i.e. cracks develop in it. As the boundary conditions of the problem change, failure beginning at some points in the volume of rock under consideration will, in general, spread to other elements. In each element, just before failure the equality of (1.1) will be attained. The set of points of the volume of rock in question at which the equality of (1.1) is reached and at which failure is about to occur forms the boundary between the intact part of the rock and the failed part.

Two types of failure propagation are possible. In one case the elements for which the equality of (1.1) is reached are located on a line, so that as that line moves through the volume of rock a macroscopic crack is formed. In other cases these elements occupy a two-dimensional surface which, as it propagates, shifts a three-dimensional volume of material from the continuous state to the failed condition. In the first case the suitable boundary conditions to be formulated on the crack surface do not coincide with (1.1) taken with the equal sign. This is the correct condition only at the edge of the crack. The condition at the edge of a crack in a brittle elastic material can be formulated somewhat differently by specifying the character of the singularity in the framework of the linear theory of elasticity. This case has been fully studied [1] and will not be considered here. In the second case, the equality of (1.1) is satisfied on the surface which separates the failed portion of rock from the intact part as a limiting condition which is approached as we take points closer and closer to the surface from the intact part. In the present case, many fine cracks are formed behind the failure surface. The failed material may, therefore, also be considered as a continuous medium; i.e. it can be described by equations of continuum mechanics.

The equality of (1.1) is a boundary condition for the intact region; it is attained on a surface which is unknown in advance and is determined in the course of the solution of the problem. The motions and deformations on the two sides of this surface are described by different systems of equations for the intact and failed regions; the equality of (1.1) is not satisfied in these regions. Only in the special case in which plastic flow occurs in the failed region and when the yield condition coincides with (1.1) is the equality of (1.1) satisfied in the region behind the failure surface as a yield condition, if plastic flow occurs t there.

In general, the material in both the intact and failed states will be described by the equations of elastic-plastic deformation of a medium. The mechanical characteristics (constants, parameters and functions) will be different in the intact and failed states. A surface separating the two states will be called a failure front. It will generally be a surface of strong discontinuity (a shock front) or a contact discontinuity.

In all cases, if the properties of the failed and intact materials are known, the natural conditions of compatibility at the failure front (the conservation laws) together with the equality of (1.1) form a system of boundary relations formulated on the surface of discontinuity (the failure front) which is sufficient for the unique solution of the problem as a whole.

Let us take a concrete example of the condition (1.1). If the principal stresses $\sigma_1, \sigma_2, \sigma_3$ in the problem can become tensile, then the following condition will be the simplest natural condition for an isotropic material

$$\sigma_i \leqslant \sigma_* \qquad (i = 1, 2, 3) \tag{1.2}$$

If the equality is attained in any of these relations, fracture occurs on the corresponding principal plane, after which the principal stress on this plane becomes zero. If, however, the principal stresses are negative, i.e. are compressive, then it is possible to consider that failure takes place by means of slip along the planes on which the maximum shear stress acts, or else on the octahedral planes. For an isotropic material the condition (1.1) reduces to the following relations corresponding to these two possible assumptions:

$$2\tau_i \equiv |\sigma_j - \sigma_k| \leqslant 2\tau_* \qquad (i, j, k = 1, 2, 3) \tag{1.3}$$

or

$$\tau_{oct} \equiv \sqrt{{}^1/_3(S_1^2 + S_2^2 + S_3^2)} = \sqrt{{}^2/_3 I_2} \leqslant \tau_*$$
$$S_i = \sigma_i + p, \qquad p = -{}^1/_3(\sigma_1 + \sigma_2 + \sigma_3) \tag{1.4}$$

In this way, the condition (1.1) is reduced to the relations (1.2) and (1.3) or to (1.2) and (1.4). The inequalities (1.2) and (1.3) define a region in the space of principal stresses $\sigma_1, \sigma_2, \sigma_3$ which is bounded by the semi-infinite hexagonal prism (1.3) and the plane (1.2). In the case of (1.4) the prism becomes a circular cylinder. If a state of stress is produced at the points of a failure front as these points are approached from the intact material which corresponds to one of the equalities of (1.2), then failure will be accompanied by the formation of tension cracks oriented along the corresponding principal planes. If, however, the state of stress corresponds to points of the prism (1.3) or the cylinder (1.4), then shear cracks will be formed on the corresponding planes. For the first case, the corresponding normal stress will be absent in the failed rock mass until the cracks close up (such closing of cracks may or may not ensue depending on the boundary conditions of the problem). Before such closing of cracks, the failed material will be described by a system of equations which are not the same as the equations for the intact rock mass. After closing of the cracks the equations will be the same.

It is also possible that closing of the cracks does not occur, or that, on the contrary, a second failure may take place which now corresponds to Eqs. (1.3) or (1.4). The equations describing the material which has failed a second time are again altered.

In the simplest case it may be assumed that the equations describing the failed material for failure in accordance with Eqs. (1.3) or (1.4) are the same independently of whether this is a first failure or one which has followed a previous failure of the type of Eq. (1.2). We shall take for these equations the elastic-plastic model for a soft soil considered in [2], beginning from the assumption that when the material fails many fine cracks are formed which break up the material into pieces which are small compared to the characteristic dimension of the problem. In this way the properties of the material approximate those of a dispersed uncemented soil. In this theory, for nonporous rock in the intact state the volumetric compressibility will be considered to be reversible and, in the simplest case, linear, The yield condition may be chosen in the usual form $I_2 = F(p)$, and in the special case when internal friction after failure is such that it can be described by the failure condition produced at the instant of failure, the yield condition can be taken the same as the failure condition, i.e. in the form (1.3) or (1.4).

The failure criterion for porous rocks in the initial state will be somewhat different. Failure of the skeleton will take place in such rocks under purely hydrostatic compression. This failure will be gradual and will occur over some range of pressure, so that the strength of the material will decrease with increasing hydrostatic pressure. It may therefore be considered that the condition (1.2) is retained, but that in the conditions (1.3) and (1.4) the quantity τ_* must be regarded as a function of p which is constant as p varies from small magnitudes up to some value p_1 at which failure of the skeleton begins, and which decreato zero as p increases from p_1 to the value p_2 at which the skeleton fails completely and the material becomes a sand.

Moreover, the volumetric compressibility of such materials in the "unfailed" state in the pressure interval $p_1 \leqslant p \leqslant p_2$, and also in the failed state, will no longer be reversible or linear, so that the failed material will be described by the equations of a soft soil [2]. Here the yield criterion cannot be taken the same as the failure criterion, since in the failure criterion (1.4) $d\tau_*/dp \leqslant 0$, while the yield criterion written in the form $I_2 = F(p)$ must satisfy the relation $dF/dp \geqslant 0$.

We remark that, generally speaking, the last case also includes all soft soils which exhibit cohesion. However, for these p_1 and p_2 are very small. Therefore, in the construction of a model of such soils the strength properties (or rather brittleness properties) can be neglected as was done in [2]. It should be kept in mind that for soft soils having cohesion, the strength resisting tensile failure can increase in the process of plastic deformation and irreversible compression, so that the condition (1.2) should be retained for these materials, assuming that σ_* depends on the amount of residual densification compaction θ_*, i.e. we should set

$$\sigma_* = \sigma_*(\theta_*) \tag{1.5}$$

Experiments investigating loess soils under the action of an explosion [3] actually exhibit the phenomenon of brittle behaviour when blast waves propagate through the soil. Sectioning of the explosion cavity after the soil motions had ceased showed a considerable volume of soil around the cavity permeated with a large number of cracks which were oriented in the radial direction [3]. This is indicative of the fact that the loess had enough cracking strength to develop considerable tensile stresses when the wave passed. After these attained the limiting value, the soil mass was broken up by radial cracks.

To complete the system of equations of the model of hard brittle rocks, specific equations are required for the intact material fissured by tensile cracks. The simplest assumption consists in the adoption of the linear equations of the theory of elasticity for the intact mass mass and some invariant of these for the mass which has undergone failure by cracking in tension. The character of this variant is determined by the fact that one of the principal stresses is zero in the failed part of the rock. Therefore, the equations of elasticity for this rock mass must be written for a smaller number of spatial dimensions; that is, equations of the type used for a state of plane stress or the equations of the theory of elastic rods.

The assumption of linearly elastic behavior of the intact material may not correspond to the experimental data (there is some experimental information on this point [4]).

However, consideration of plasticity in the intact mass will, in general, introduce no essential difficulties and can be included in solving problems. This will be shown below when a specific problem is considered.

We remark that the model of a brittle material which has been constructed is one in which the energy, from a thermodynamic point of view, is decoupled into a mechanical and a thermal part. The thermodynamic correctness of the theory can be established in the same way as was done for the model of a soft soil. An additional factor which should be considered here in the treatment of the energy equation is the energy expended in the formation of the crack surfaces (the surface energy of the cracks). Later on, in the examination of a specific problem, an actual estimate of the role played by surface energy in the failure process will be made.

The model which has been constructed here, like the model for a soft soil, is a limiting case for the consideration of rapid processes (explosions and shocks) and for static processes. It does not contain constants from which quantities with the dimensions of length or time can be formed. It therefore admits of the same similarity conditions and rules of scaling as the model for a soft soil (see [2]).

Experience in the blasting of ledge rock [5 and 6] reveals that this similarity rule actually holds. Therefore, in this respect, the proposed model corresponds to the real properties of hard rocks.

In conclusion, we remark that the original motivation for the theory which has been outlined above was prompted by the experiments of Adushkin and Sukhotin [7], our experiments with loess [3], and also the contributions [8 and 10] in which special cases of material which fails in a brittle manner were considered. The theory developed in this paper is a natural generalization of the model for a soft soil [2] and of the results of the references quoted.

2. Let us consider a spherical cavity of radius r_0 in a space filled with a homogeneous, isotropic rock which is at rest and is compressed by a hydrostatic pressure p_0. The cavity$_0$ is filled with an explosive charge which becomes a gas having an initial pressure p_{00} after detonation. Because of this pressure the rock surrounding the cavity is set into motion, as a result of which, if p_{00} is large enough, a part of the rock will fail. However, at large distances from the cavity the stress waves will decay.

For the various requirements of mining, seismic prospecting and other applications, it is of interest to determine the volume of rock which fails, the character of the failure, the parameters of the waves emanating from the focus of the explosion, etc. as functions of the properties of the rock, the explosive charge and the initial stresses in the rock. This problem has been studied in schematic form in the papers [8 and 9] and in [10] for the static case when the pressure in the cavity increases slowly from a value equal to the initial pressure in the rock. A formulation of the problem is presented below, based on the mathematical model developed in Section 1. Under certain natural assumptions about the properties of the medium, the problem reduces to an initial value problem for a system of ordinary differential equations.

We shall consider that the deformed material in the intact state is governed by Hooke's law

$$\sigma_r = -p_0 + \lambda\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) + 2\mu\frac{\partial u}{\partial r}$$
$$\sigma_\theta = -p_0 + \lambda\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) + 2\mu\frac{u}{r}, \qquad \sigma_\varphi = \sigma_\theta \tag{2.1}$$

where λ and μ are the Lamé constants; r is a Lagrangian coordinate; u is the radial displacement; σ_r, σ_θ, and σ_φ are the stresses referred to spherical coordinates. These are principal stresses by virtue of the symmetry of the problem.

The equation of motion has the form

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma_r}{\partial r} + \frac{2}{r}(\sigma_r - \sigma_\theta) \tag{2.2}$$

where ρ is the density of the rock.

Substitution of the relation (2.1) into (2.2) and the introduction of the dimensionless variables

$$\tau = \frac{c_0 t}{r_0}, \quad x = \frac{r}{r_0}, \quad U = \frac{u}{r_0}, \quad \Sigma_r = \frac{\sigma_r}{\rho c_0^2},$$
$$\Sigma_\theta = \frac{\sigma_\theta}{\rho c_0^2}, \quad \rho c_0^2 = \lambda + 2\mu \tag{2.3}$$

reduces Eq. (2.2) to the form

$$\frac{\partial^2 U}{\partial \tau^2} = \frac{\partial^2 U}{\partial x^2} + \frac{2}{x}\frac{\partial U}{\partial x} - \frac{2}{x^2}U \tag{2.4}$$

The solution of this equation corresponding to a diverging wave is given by Formula

$$U = \frac{f'(\tau - x)}{x} + \frac{f(\tau - x)}{x^2} \tag{2.5}$$

where $f(\xi)$ is an as yet arbitrary function of its argument.

Substituting (2.5) into (2.1) and (2.3) we obtain for the stresses and displacement(*)

$$\sigma_r = -\rho c_0^2 \left\{ \frac{1}{x} f''(\tau - x) + 4\gamma^2 \left[\frac{1}{x^2} f'(\tau - x) + \frac{1}{x^3} f(\tau - x) \right] \right\} - p_0$$

$$\sigma_\theta = -\rho c_0^2 \left\{ \frac{1 - 2\gamma^2}{x} f''(\tau - x) - 2\gamma^2 \left[\frac{1}{x^2} f'(\tau - x) + \frac{1}{x^3} f(\tau - x) \right] \right\} - p_0$$

$$u = r_0 \left[\frac{1}{x} f'(\tau - x) + \frac{1}{x^2} f(\tau - x) \right] \qquad (2.6)$$

If p_{00} is not very large, failure will not occur. In view of the smallness of the displacements the cavity expands only slightly with time and the pressure in it does not vary significantly. In this case the condition which determines the function $f(\xi)$ in (2.6) is written in the form

$$\sigma_r \big|_{r=r_0} = -p_{00} \qquad (2.7)$$

Substituting the first of Eqs. (2.6) into (2.7), we obtain a differential equation for $f(\xi)$. Its solution for the natural initial data

$$f(-1) = 0, \qquad f'(-1) = 0 \qquad (2.8)$$

which follow from the initial condition

$$u(r, 0) = 0 \qquad (2.9)$$

has the form

$$f(\xi) = \frac{P}{4\gamma^2} \left\{ 1 - \frac{\exp[-2\gamma^2(\xi + 1)]}{\sqrt{1 - \gamma^2}} \sin[2\gamma \sqrt{1 - \gamma^2}(\xi + 1) + \varphi] \right\} \qquad (2.10)$$

$$\sin\varphi = \sqrt{1 - \gamma^2}, \qquad \cos\varphi = \gamma, \qquad P \equiv (p_{00} - p_0)/\rho c_0^2$$

In order to establish the conditions under which failure occurs, we now turn to the failure criterion. In the present case, by virtue of the symmetry of the stress situation there are only two essentially different stress components σ_r and σ_θ. The failure criterion is therefore formulated in the form of a relation between these quantities. It is convenient to represent the failure criterion graphically. In Fig. 1 the straight-line segments BC and B_1C correspond to the condition (1.2). The lines BA and B_1A_1 correspond to the condition (1.3). If the condition (1.4) is used, these lines are displaced somewhat. However, the displacement is very slight (instead of $-2\tau_*$, the intercepts on the axes will be $-3\tau_*/\sqrt{2}$, which differs from $-2\tau_*$ by 6%). Therefore, in the present problem, the adoption of (1.3) or (1.4) gives practically identical results. From the solution of the elastic problem (2.6) and (2.10), we have

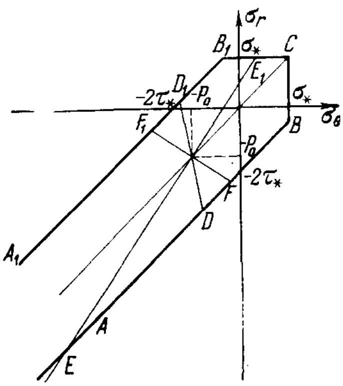

Fig. 1

$$\sigma_r \Big|_{\substack{\tau=0 \\ x=1}} \equiv \sigma_{r_0} = -p_{00} < 0$$

$$\sigma_\theta \Big|_{\substack{\tau=0 \\ x=1}} \equiv \sigma_{\theta_0} = -(1 - 2\gamma^2) p_{00} - 2\gamma^2 p_0 < 0 \qquad (2.11)$$

It is clear from this that under the condition

$$p_{00} \leqslant p_0 + \gamma^{-2}\tau_* \qquad (2.12)$$

failure will not occur at the initial time, when motion begins.

If, however, the inequality (2.12) is violated, then failure by the formation of shear cracks will take place around the explosion cavity at the very start of the motion of the rock. It is

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*) γ is the ratio of the S-wave velocity to the S-wave velocity; $\gamma^2 = \mu/(\lambda + 2\mu)$. (Translator's note)

then necessary to construct the solution for the failed material in the vicinity of the cavity and to join it up to a solution of the type (2.6) at the failure front, which, by virtue of the symmetry of the problem, will be an expanding sphere.

We shall first consider the case in which the condition (2.12) is satisfied and we shall explain how the motion which is described at short time by Eqs. (2.6) and (2.10) develops for later times. The stress σ_r on the cavity boundary does not change with time; it remains equal to p_{00}. The stress σ_θ at the same place varies with time according to the formula

$$\sigma_\theta|_{x=1} = 2\gamma^2(p_{00}-p_0) - p_{00} + \\ + \frac{3-4\gamma^2}{2}(p_{00}-p_0)\left[1 - \frac{\exp(-2\gamma^2\tau)}{\sqrt{1-\gamma^2}}\sin\left(2\gamma\sqrt{1-\gamma^2}\,\tau + \varphi + \psi\right)\right] \quad (2.13)$$

$$\sin\psi = 2\gamma\sqrt{1-\gamma^2}, \qquad \cos\psi = 1 - 2\gamma^2$$

It is clear from this expression that the stress σ_θ decays exponentially and oscillates about the limiting equilibrium value

$$\sigma_{\theta\infty} \equiv \sigma_\theta(1, \infty) = -p_{00} + {}^3/_2(p_{00} - p_0) \quad (2.14)$$

For ease of visualization of the arguments which follow, we shall represent the set of initial values of σ_r and σ_θ given by Eqs. (2.11) and the limiting values given by Eq. (2.14) and by $\sigma_{r\infty} = \sigma_{r0} = -p_{00}$ on the $\sigma_r - \sigma_\theta$ plane for fixed p_0 and variable p_{00}. In Fig. 1, the straight line EE_1 represents the initial state, DD_1 the limiting state. It is obvious that for $p_{00} > p_0$ we have that $\sigma_{\theta\infty} > \sigma_{\theta 0}$, and for $p_{00} < p_0$, $\sigma_{\theta\infty} < \sigma_{\theta 0}$. Therefore, if the condition (2.12(and the condition $\sigma_{r0} = -p_{00} < \sigma_*$ are satisfied, i.e. if the initial point is within the region $ABCB_1A_1$, then for $p_{00} > p_0$ the stress $\sigma_\theta(1, \tau)$ increases with time initially, it decreases initially for $p_{00} < p_0$. In the first case, it is possible to reach the boundary ABC of the failed state, and failure can occur by shear (on the segment BA) or by tension cracking (on the segment BC). In the second case failure is possible only by means of shear. In the first case the maximum value of $\sigma_\theta(1, \tau)$ (and in the second case its minimum value) will be reached when $\partial\sigma_\theta(1, \tau)/\partial\tau$ first equals zero. If for these values, the point σ_r, σ_θ remains inside the region $ABCB_1A_1$ and failure does not occur on the cavity boundary. If, however, the values correspond to a point beyond this region then at some time preceding the attainment of $\max\sigma_\theta$ or $\min\sigma_\theta$, failure begins at the cavity boundary.

The derivative $\partial\sigma_\theta(1, \tau)/\partial\tau$ first becomes zero for

$$2\gamma\sqrt{1-\gamma^2}\,\tau_1 + \psi = \pi \quad (2.15)$$

The corresponding value of σ_θ is

$$\sigma_{\theta 1} \equiv \sigma_\theta(1, \tau_1) = \\ = -(1-2\gamma^2)\,p_{00} - 2\gamma^2 p_0 + {}^1/_2(3-4\gamma^2)(p_{00}-p_0)(1 + e^{-2\gamma^2\tau_1}) \quad (2.16)$$

For $p_{00} - p_0 > 0$, we have $\sigma_{\theta 1} > \sigma_{\theta 0}$, i.e. $\sigma_{\theta 1} = \max\sigma_\theta(1, \tau)$, and for $p_{00} - p_0 < 0$, $\sigma_{\theta 1} < \sigma_{\theta 0}$, i.e. $\sigma_{\theta 1} = \min\sigma_\theta(1, \tau)$. The locus of the points $\sigma_r = \sigma_{r1} = \sigma_{r0} = -p_{00}$, $\sigma_\theta = \sigma_{\theta 1}$ for fixed p_0 and variable p_{00} is shown on Fig. 1 as the straight line FF_1.

If p_{00} is such that the point σ_{r1}, $\sigma_{\theta 1}$ is located on the segment FF_1 failure does not occur and the solution is completely described by Eqs. (2.6) and (2.10). If, however, $p_0 + \gamma^{-2}\tau_* > p_{00} > p_{00F}$ or $-\sigma_* < p_{00} < p_{00F_1}$ (where p_{00F} and p_{00F_1} are the values of p_{00} which correspond to the point σ_{r1}, $\sigma_{\theta 1}$ in the locations F and F_1), then beginning at a time $\tau_2 < \tau_y$ where $\tau_2 = \tau_2(p_{00})$ is the instant at which the point σ_r, σ_θ moves onto the boundary of the region $ABCB_1A_1$, failure starts on the cavity boundary and any further consideration of the solution must take the failure into account. We find the value of p_{00F_1} from the relation

$$\sigma_{r_1} - \sigma_{\theta_1} = 2\tau_* \quad (2.17)$$

The value of p_{00F} is determined in two different ways depending on whether the point F is located on AB or on BC. In the first case, p_{00F} is found from the relation

$$\sigma_{r_1} - \sigma_{\theta_1} = -2\tau_* \quad (2.18)$$

and in the second case from the relation

$$\sigma_{\theta_1} = \sigma_* \tag{2.19}$$

From Eqs. (2.17) and (2.18), we have

$$p_{00F_1} = p_0 - \frac{\tau_*}{\gamma^2 + (^3/_4 - \gamma^2)\,[(1 + \exp(-2\gamma^2\tau_1)]} \tag{2.20}$$

$$p_{00F} = p_0 + \frac{\tau_*}{\gamma^2 + (^3/_4 - \gamma^2)\,[(1 + \exp(-2\gamma^2\tau_1)]} \tag{2.21}$$

We obtain from Eq. (2.19) that

$$p_{00F} = p_0 + \frac{2(\sigma_* + p_0)}{1 + (3 - 4\gamma^3)\exp(-2\gamma^2\tau_1)} \tag{2.22}$$

The value $p_0 = p_0^*$ which separates the regions of values corresponding to Eq. (2.21) and to (2.22) is obtained by equating the right-hand sides of Eqs. (2.21) and (2.22):

$$p_0^* = -\sigma_* + 2\tau_* \frac{1 + (3 - 4\gamma^2)\exp(-2\gamma^2\tau_1)}{3 + (3 - 4\gamma^2)\exp(-2\gamma^2\tau_1)} \tag{2.23}$$

Finally, we note the value $p_0 = p_0^{**}$ obtained from (2.20) for $p_{00F_1} = -\sigma_*$ (for this value, the point F_1 is at B)

$$p_0^{**} = -\sigma_* + 2\tau_* \frac{2}{3 + (3 - 4\gamma^2)\exp(-2\gamma^2\tau_1)} \tag{2.24}$$

If $p_0 < p_0^{**}$, the point F_1 falls on the segment B_1C, but this means that for $-\sigma_* < p_{00} < p_0$ failure does not occur at the cavity, and only for $p_0 = -\sigma_*$ are the particles which actually form the cavity wall torn away from the rock mass. The stress there then falls to zero, so that the motion that occurs corresponds to an initial value of pressure equal to $p_{00} = 0$. The value of σ_{θ_1} which corresponds to this value of p_{00}, for $p_0 = p_0^{**}$, is

$$\sigma_{\theta_1}^{**} = {}^1/_2\, \sigma_* (3 + a) - 2\tau_*, \qquad a \equiv (3 - 4\gamma^2)\exp(-2\gamma^2\tau_1) \tag{2.25}$$

Since $\sigma_{\theta_1}^{**} > -2\tau_*$, the point $\sigma_{r1} = -p_{00} = 0$, $\sigma_{\theta_1} = \sigma_{\theta_1}^{**}$ is located above the line AB In this case, if $\sigma_{\theta_1}^{**} < \sigma_*$, the motion which occurs does not lead to failure at the cavity. If, however, $\sigma_{\theta_1}^{**} > \sigma_*$, then at some instant failure begins at the boundary by the formation of radially oriented tension cracks. Such a failure will also take place in the case $\sigma_{\theta_1}^{**} < \sigma_*$ beginning with some value of $p_0 < p_0^{**}$.

These cases exhaust all the possible situations arising from all possible specifications of the parameters p_{00}, p_0, γ, σ_*, τ_*. We note that although the case in which $p_{00} < p_0$ is of no interest for the problem of an explosion, the qualitative peculiarities in this case are quite curious. Indeed, if $p_0 > p_0^{**}$, then in this case the removal of the initial stress on the cavity surface can lead to shear failure. However, if $p_0 < p_0^{**}$, then a tensile spherical crack may occur at the initial time, after which a failure front causing radial tensile cracks to form may propagate outside this spherical crack.

We also note the following. For static loading, failure obviously occurs only if the point σ_r, σ_θ which moves along the segment DD_1 reaches an end point of the segment. Since the point F, which corresponds to the occurrence of failure under dynamic conditions, is located above point D on AB, and the point F_1 is below D_1 on A_1B_1, then under dynamic conditions the capability of the medium to withstand pressures applied to the cavity is lower than under static conditions for $p_{00} > p_0$, and is higher for $p_{00} < p_0$. The situation is analogous in the case when both the points D and F, or only F, lie on the segment BC, or when D_1 lies on B_1C and F_1 remains on A_1B_1. When the points D_1 and F_1 both fall on B_1C then the static and dynamic strengths turn out to be the same.

The question of the possibility of a failure occurring at the surface of the cavity has been studied in the preceding analysis. However, generally speaking, failure can also occur far from the cavity. It is therefore necessary to establish where the point $(\sigma_r, \sigma_\theta)$ determined from the solution (2.6) and (2.10) passes over the boundary $ABCB_1A_1$. To solve this problem in its general form is very difficult, but it is possible to formulate a procedure for the solution which can be carried out numerically in each particular case.

Failure can begin at some internal point of the region

$$x > 1, \quad \xi \equiv \tau - x > -1$$

if one of the following inequalities is satisfied there

$$\sigma_\theta(x, \xi) = \sigma_*, \ \sigma_r(x, \xi) = \sigma_*, \ \sigma_r(x, \xi) - \sigma_\theta(x, \xi) = \pm 2\tau_*$$

By computing the derivatives with respect to x of the left-hand sides of these expressions for ξ = const, and setting these equal to zero, we obtain quadratic equations for x.

Thus the quantities $\sigma_\theta(x, \xi)$, $\sigma_r(x, \xi)$, and $\sigma_r(x, \xi) - \sigma_\theta(x, \xi)$ have, in general, two extrema on each characteristic ξ = const and tend to $-p_0$ or to zero as $x \to \infty$. Therefore, if for a given ξ failure is not reached on the cavity boundary $x = 1$, then it will be reached first at one of these two extrema. It is possible to write out explicit formulas for these extrema. There is, therefore, no difficulty in calculating their values in any particular case and in finding for all values of ξ the point at which failure begins. The extremal points are determined by Formulas

$$x_{1,2} = \frac{2\gamma^2 f'(\xi) \pm \sqrt{4\gamma^2 [f'(\xi)]^2 + 6\gamma^2(1-2\gamma^2) f(\xi) f''(\xi)}}{(1-2\gamma^2) f''(\xi)} \quad \text{for } \sigma_\theta \tag{2.26}$$

$$x_{1,2} = 4\gamma^2 \frac{-f'(\xi) \pm \sqrt{[f'(\xi)]^2 - f(\xi) f''(\xi)}}{f''(\xi)} \quad \text{for } \sigma_r \tag{2.27}$$

$$x_{1,2} = 3\frac{-f'(\xi) \pm \sqrt{[f'(\xi)]^2 - f(\xi) f''(\xi)}}{f''(\xi)} \quad \text{for } \sigma_r - \sigma_\theta \tag{2.28}$$

For $\xi < \xi_1$ let the solution (2.6), (2.10) nowhere attain a failure condition, and let the condition determined by one of the Eqs. (2.26) to (2.28) be satisfied for the first time for $\xi = \xi_1$ and $x = x_1$. It is then obvious that for nearby values of $\xi > \xi_1$ an entire segment of each characteristic ξ = const will correspond to points σ_r, σ_θ which fall outside the region $ABCB_1A_1$, and that the failure condition will be satisfied at the ends of these segments.

The set of these end points forms a line passing through the point $\xi = \xi_1$, $x = x_1$ which remains in the region $\xi > \xi_1$ and is tangent to the characteristic $\xi = \xi_1$ at this point. On this line there can exist a point $x = x_2 > 1$ in the region $x < x_1$ at which the line is first tangent to a characteristic of the second family of Eqs. (2.4), i.e., to the line $\tau + x = \text{const} = \xi_2 - 2x_2$. The part of this line located between the two points x_1 and x_2 is the true failure front, because, in the first place, the failure condition is satisfied as this line is approached from the intact region and, secondly, at each point of the line the characteristics of both families drawn backwards, i.e., to the x-axis, are located on one side of the line (the line is space-like). Therefore, the solution of the problem at points of this line as they are approached from below (from the side on which the x-axis is located) is uniquely determined by Eqs. (2.6). The continuation of the solution beyond the end points at which the line is tangent to the characteristics of Eq. (2.4) and where it is no longer determined by Eqs. (2.6) will be discussed somewhat later.

The further investigation of the solution of the problem will be carried out in the following way. First the solution will be constructed only for cases in which failure begins at the cavity after which, by using the relations (2.26) to (2.28) and the reasoning related to them, it will be possible to determine whether independent origins of failure occur outside the cavity. If these do occur, then until the failure fronts emanating from these origins begin to interact with the solution constructed without taking account of their existence (the nature of this interaction will be described in detail later), the original solution will be valid. From the time that the interaction starts, the procedure for continuation of the solution becomes more complicated.

Let us begin with the case in which failure by the formation of radial cracks is initiated at the cavity, i.e., when the point F falls on the segment BC and p_{00} lies in the range $p_{00F} < p_{00} < p_{00B}$, where p_{00B} is the value of p_{00} corresponding to the point B. The time τ_2 at which the failure begins is determined from the condition $\sigma_\theta(1, \tau_2) = \sigma_*$. From that time on, a spherical failure front propagates into the medium. The variation of the radius of this front with time, $x = x_1(\tau)$ is to be determined.

In the region $x > x_1(\tau)$ the solution is given by Eqs. (2.6) as before. However, in these relations the function $f(\xi)$ is no longer determined by the relation (2.10), but is found from the condition of matching of the solution (2.6) with the solution in the region $1 \leqslant x \leqslant x_1(\tau)$ which will now be determined.

Assuming that in the region $1 \leqslant x \leqslant x_1(\tau)$ a large number of fine cracks oriented in the radial direction are formed when the failure front passes, we shall consider that the motion in that region can be described by equations of a continuum. The equation of motion will have the form

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma_r}{\partial r} + \frac{2}{r} \sigma_r \tag{2.29}$$

since $\sigma_\theta = 0$ there. We obtain a second equation in the form of Hooke's law for the "bars" into which the material is broken up

$$\sigma_r = E \frac{\partial u}{\partial r} + \sigma_{r_0}, \qquad \sigma_{r_0} = -\frac{\gamma^2}{1-\gamma^2} p_0 \tag{2.30}$$

where E is Young's modulus

$$E = (\lambda + 2\mu) \frac{\gamma^2 (3 - 4\gamma^2)}{1-\gamma^2} = (\lambda + 2\mu) \frac{(1+\sigma)(1-2\sigma)}{1-\sigma} \tag{2.31}$$

and σ is Poisson's ratio.

Equation (2.30) is obtained from Hooke's law in its full form

$$\sigma_r = \lambda (\varepsilon_r + 2\varepsilon_\theta) + 2\mu \varepsilon_r - p_0, \qquad \sigma_\theta = \lambda (\varepsilon_r + 2\varepsilon_\theta) + 2\mu \varepsilon_\theta - p_0$$

under the condition $\sigma_\theta = 0$ by eliminating the strain ε_θ which is, of course, no longer equal to u/r. Substituting (2.30) into (2.29) and using the dimensionless variables of (2.3), we obtain

$$\frac{\partial^2 U}{\partial \tau^2} = q^2 \left(\frac{\partial^2 U}{\partial x^2} + \frac{2}{x} \frac{\partial U}{\partial x} \right) - \frac{2\gamma^2}{(1-\gamma^2) x} P_0 \tag{2.32}$$

$$q = \frac{c_1}{c_0} = \left(\frac{\gamma^2 (3 - 4\gamma^2)}{1-\gamma^2} \right)^{1/2} = \left(\frac{(1+\sigma)(1-2\sigma)}{1-\sigma} \right)^{1/2}, \qquad P_0 = \frac{p_0}{\rho c_0^2} \tag{2.33}$$

where c_1 is the acoustic bar velocity ($\rho c_1^2 = E$). The general solution of (2.32) is

$$U = \frac{1}{x} [f_1(q\tau - x) + f_2(q\tau + x)] + \frac{P_0}{3 - 4\gamma^2} x \tag{2.34}$$

where $f_1(\xi_1)$ and $f_2(\xi_2)$ are arbitrary functions of their arguments.

The radial stress σ_r in the failed region is (2.35)

$$\sigma_r = -\rho c_0^2 \frac{\gamma^2 (3 - 4\gamma^2)}{1-\gamma^2} \left[\frac{f_1'(q\tau - x) - f_2'(q\tau + x)}{x} + \frac{f_1(q\tau - x) + f_2(q\tau + x)]}{x^2} \right]$$

Thus, for times $\tau > \tau_2$, the solution in the intact material will be described by Eqs. (2.6) and in the failed material by Eqs. (2.34) and (2.35). Three undetermined functions f, f_1 and f_2 are contained in these equations. They must be found from the conditions of continuity at the failure front $x = x_1(\tau)$, where $x_1(\tau)$ is also an unknown function which is to be determined, and also from the conditions at the cavity.

The continuity conditions consist of the usual conservation laws on a surface of strong discontinuity and the failure condition which must be satisfied on the side of the surface toward the intact material. This last condition is written in the form

$$\frac{1-2\gamma^2}{x_1} f''(\xi) - 2\gamma^2 \left[\frac{f'(\xi)}{x_1^2} + \frac{f(\xi)}{x_1^3} \right] = -\Sigma_* - P_0, \quad \Sigma_* = \frac{\sigma_*}{\rho c_0^2} \tag{2.36}$$

As is usual when using Lagrangian coordinates, the law of conservation of mass follows automatically from the natural condition of continuity of displacements and need not be written out. However, it is necessary to write down the condition of continuity of displacement which has the form

$$f'(\xi)+\frac{1}{x_1}f(\xi)=f_1(\xi_1)+f_2(\xi_2)+\frac{P_0}{3-4\gamma^2}x_1^2$$

$$\xi_1=q\tau-x_1(\tau),\quad \xi_2=q\tau+x_2(\tau) \tag{2.37}$$

The impulse-momentum theorem is written in the form

$$-\sigma_{r_1}+\sigma_{r_2}=\rho\frac{dr_1}{dt}(v_1-v_2) \tag{2.38}$$

or, after substitution of (2.5), (2.6), (2.34) and (2.35),

$$f''(\xi)+4\gamma^2\left[\frac{1}{x_1}f'(\xi)+\frac{1}{x_1^2}f(\xi)\right]-$$
$$-\frac{\gamma^2(3-4\gamma^2)}{1-\gamma^2}\left\{f_1'(\xi_1)-f_2'(\xi_2)+\frac{1}{x_1}[f_1(\xi_1)+f_2(\xi_2)]\right\}+P_0x_1=$$
$$=x_1\left\{f''(\xi)+\frac{1}{x_1}f'(\xi)-q[f_1'(\xi_1)+f_2'(\xi_2)]\right\} \tag{2.39}$$

Turning now to the formulation of the condition at the cavity, we shall make one remark. We adopt the usual assumption of adiabatic, quasi-steady change of state of the detonation products of the explosive as the volume of the cavity changes(*). The adiabat can be approximated by two relations of the form $pV^\varkappa$ = const for the two ranges of variation of the pressure the exponent $\varkappa$ is close to 3. Therefore, although in the propagation of the failure front at a considerable distance the motion of the cavity wall will be small, the change in pressure in the cavity may be considerable because of the large size of $\varkappa$, and these changes should be taken into account in the boundary condition at the cavity.

With this in mind, we write down the condition at the cavity in the form

$$\frac{\gamma^2(3-4\gamma^2)}{1-\gamma^2}[f_1'(\xi_1^0)-f_2'(\xi_2^0)+f_1(\xi_1^0)+f_2(\xi_2^0)]=$$
$$=P_{00}\left[1+f_1(\xi_1^0)+f_2(\xi_2^0)+\frac{P_0}{3-4\gamma^2}\right]^{-3\varkappa}\approx$$
$$\approx P_{00}\left\{1-3\varkappa\left[f_1(\xi_1^0)+f_2(\xi_2^0)+\frac{P_0}{3-4\gamma^2}\right]\right\}$$
$$P_{00}=\frac{p_{00}}{\rho c_0^2},\quad \xi_1^0=q\tau-1,\quad \xi_2^0=q\tau+1 \tag{2.40}$$

The relations (2.36), (2.37), (2.39) and (2.40) form a complete system of equations for the determination of the four functions $f(\xi)$, $f_1(\xi_1)$, $f_2(\xi_2)$ and $x_1(\tau)$. We note that the stress σ_r and, therefore, the velocity v at the failure front are not continuous, for the addition of one of these continuity conditions would overdetermine the system (2.36) to (2.39) and make the problem insoluble. The failure front is a real shock wave. Therefore, the continuity condition on σ_r adopted in the articles [8 and 9] is incorrect in the general case.

Let us now examine the relations at the failure front in greater detail. Differentiating the condition of continuity of displacement, we obtain

$$v_1-v_2=\frac{dr_1}{dt}\left(\frac{\partial u_2}{\partial r}-\frac{\partial u_1}{\partial r}\right) \tag{2.41}$$

The subscripts 1 and 2 denote quantities in the intact and failed materials, respectively, Substituting (2.41) into (2.38), we obtain

$$\left|\frac{dr_1}{dt}\right|=\left(\frac{-\sigma_{r_1}+\sigma_{r_2}}{\rho(\partial u_2/\partial r-\partial u_1/\partial r)}\right)^{1/2} \tag{2.42}$$

Then, by eliminating the quantity u/r from (2.1) for $\sigma_\theta=\sigma_*$, we find

*) This condition is quite a good approximation in the case of soft soils [11]. In the present case, when the acoustic velocity is of the order of 5000 meters/sec, and the failure process involves a volume of rock extending to several charge radii, the applicability of this condition is not obvious, and wave motions in the detonation products may prove to be important.

$$\sigma_{r_1} = E\frac{\partial u_1}{\partial r} + \frac{1-2\gamma^2}{1-\gamma^2}\sigma_* - \frac{\gamma^2}{1-\gamma^2}p_0 \tag{2.43}$$

and using (2.30), we have for σ_{r_2}:

$$-\sigma_{r_1} + \sigma_{r_2} = \rho c_1^2\left(\frac{\partial u_2}{\partial r} - \frac{\partial u_1}{\partial r}\right) - \frac{1-2\gamma^2}{1-\gamma^2}\sigma_* \tag{2.44}$$

From the equations which have been obtained, it is clear in particular that if $v_1 - v_2 = 0$ then also $\partial u_1/\partial r - \partial u_2/\partial r = 0$, which is possible only for $\sigma_* = 0$. In all other cases the failure front is a shock wave. We obtain from (2.42) and (2.44) that

$$\rho\left[c_1^2 - \left(\frac{dr_1}{dt}\right)^2\right] = \frac{1-2\gamma^2}{1-\gamma^2}\,\frac{\sigma_*}{\partial u_2/\partial r - \partial u_1/\partial r} \tag{2.45}$$

If $\partial u_2/\partial r - \partial u_1/\partial r > 0$, then it follows from (2.42) that $-\sigma_{r_1} > -\sigma_{r_2}$, and since the change in density is determined by Formula

$$\frac{\Delta\rho}{\rho} = -\frac{\partial u}{\partial r} - \frac{2u}{r}$$

and at the failure front

$$\frac{\Delta\rho_1 - \Delta\rho_2}{\rho} = \frac{\partial u_2}{\partial r} - \frac{\partial u_1}{\partial r} \tag{2.46}$$

then $\Delta\rho_1 > \Delta\rho_2$ also; i.e. the failure front is a rarefaction front. It should be recalled that $\Delta\rho_2$ is the change in mean density of a material which has failed by radial cracking, and $\Delta\rho_1$ is the change of true density of a continuous material. The change of true density during failure can be determined in accordance with the discontinuity in mean stress.

It follows from Eq. (2.45) that in this case the inequality $|dr_1/dt| < c_1$ will also be satisfied; i.e., the failure front propagates with subseismic velocity relative to the failed material, and *a fortiori* relative to the intact material, since $c_1 < c_0$. In this case, the front radiates elastic waves into the intact material. This is the fundamental case.

If, however, $\partial u_2/\partial r - \partial u_1/\partial r < 0$, then $-\sigma_{r_1} < -\sigma_{r_2}$, $\Delta\rho_1 < \Delta\rho_2$ and $|dr_1/dt| > c_1$. This case can occur when the failure originates in the region $x > 1$ and the law of motion of the failure front is at first determined completely by the elastic wave in the intact material, the inequality $|dr_1/dt| > c_0$ also holding. In this case the relation $r = r_1(t)$ is known from the elastic solution and the parameters of the motion right at the failure front, i.e., the functions $f_1(\xi_1)$ and $f_2(\xi_2)$ are determined completely by the relations on this front, i.e., by Eqs. (2.37) and (2.39) (*v. supra*). It should be noted that this mathematically possible case is physically admissible if the true transverse strain ε_θ° behind the failure front does not exceed the geometric (fictitious) strain, the quantity u/r, for only in this case can radial cracking occur physically. Therefore, in this case a condition which reduces to the following inequality must be satisfied:

$$\frac{U_1}{x_1} + \frac{1-2\gamma^2}{2(1-\gamma^2)}\,\frac{\partial U_2}{\partial x} - \frac{P_0}{2(1-\gamma^2)} \geqslant 0 \tag{2.47}$$

If this inequality is not satisfied, then starting at the point where it is first violated, the law of motion $x = x_1(\tau)$ must be constructed differently, in such a way that an elastic wave is radiated by the front $x = x_1(\tau)$, but does not by itself determine the law of motion of the front.

The case in which $c_1 < |dr_1/dt| < c_0$ is not possible. In this case, the line $r = r_1(t)$ is a space-like manifold for Eq. (2.32); i.e., from each point of this line the characteristics of both families fall in the region where τ increases. An analogous situation obtains with regard to the characteristics $\tau - x = \text{const}$ and $\tau + x = \text{const}$ of Eq. (2.4) for the cases $dr_1/dt > 0$ or $dr_1/dt < 0$, respectively. Therefore, the solution of the problem in both the regions of failed and intact material becomes non-unique (see the analogous situation in the problem of an explosion in soft soil [11]). This means that at the time when the state $|dr_1/dt| = c_0$ is reached (if, of course, the condition (2.47) is satisfied up to that time), the velocity of the front must change discontinuously, becoming less than c_1 in absolute value. If the condition

(2.47) is violated earlier, this velocity change of the front must take place at the time when the condition is violated.

3. Before going on to the analysis of the system of functional-differential equations (2.36), (2.37), (2.39) and (2.40) and to the construction of an algorithm for its solution, we shall examine the energy equation for the failure front. As is usual, this equation can be reduced to a chock adiabat (Hugoniot) having the form

$$\varepsilon_1 - \varepsilon_2 - \frac{1}{2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)(\sigma_{r_1} + \sigma_{r_2}) = 0 \tag{3.1}$$

where ε_1 and ε_2 are the internal energy per unit mass of the intact and failed material, respectively. By virtue of the fact that the adiabatic and isothermal elastic moduli of solids differ only slightly as a result of the smallness of the coefficient of thermal expansion, they can be considered as identical and the internal energy is separable [12]. This gives

$$\varepsilon_1 = \frac{p_1^2}{2(\lambda + {}^2/_3\mu)\rho} + \frac{J_{21}}{2\mu\rho} + \varepsilon_0(T_1) \qquad \left(p_1 = -\frac{\sigma_{r_1} + 2\sigma_*}{3}\right) \tag{3.2}$$

$$\varepsilon_2 = \frac{\sigma_{r_2}^2}{2E\rho} + \varepsilon_0(T_2) + \frac{1}{\rho}\varepsilon_s \qquad \left(J_{21} = \frac{(\sigma_{r_1} - \sigma_*)^2}{3}\right) \tag{3.3}$$

where $\varepsilon_0(T)$ is an additive part of the internal energy depending only on the temperature T. The quantity ε_s is the total surface energy of all the cracks contained in a unit volume of the failed rock. It is proportional to the total surface area of these cracks. Substituting (3.2) and (3.3) into (3.1) and using the expressions for $1/\rho_1$ and $1/\rho_2$ in terms of u_1 and u_2, we obtain

$$\frac{(\sigma_{r_1} + 2\sigma_*)^2}{18(\lambda + {}^2/_3\mu)} + \frac{(\sigma_{r_1} - \sigma_*)^2}{6\mu} - \frac{\sigma_{r_2}^2}{2E} + \rho\left[\varepsilon_0(T_1) - \varepsilon_0(T_2)\right] -$$
$$- \varepsilon_s - \frac{1}{2}(\sigma_{r_1} + \sigma_{r_2})\left(\frac{\partial u_1}{\partial r} - \frac{\partial u_2}{\partial r}\right) = 0 \tag{3.4}$$

Eliminating the difference $\partial u_1/\partial r - \partial u_2/\partial r$ with the aid of (2.44), we obtain

$$\frac{(\sigma_{r_1} + 2\sigma_*)^2}{18(\lambda + {}^2/_3\mu)} + \frac{(\sigma_{r_1} - \sigma_*)^2}{6\mu} - \frac{\sigma_{r_1}^2}{2E} + \frac{\sigma_{r_1}^2 - \sigma_{r_2}^2}{2E} -$$
$$- \frac{1}{2E}(\sigma_{r_1} + \sigma_{r_2})\left(\sigma_{r_1} - \sigma_{r_2} - \frac{\lambda}{\lambda + \mu}\sigma_*\right) + \rho\left[\varepsilon_0(T_1) - \varepsilon_0(T_2)\right] - \varepsilon_S = 0 \tag{3.5}$$

After carrying out some transformations, we find from this

$$\frac{\sigma_*}{\rho c_0^2 2\gamma^2(3 - 4\gamma^2)}\left[\sigma_* - (1 - 2\gamma^2)(\sigma_{r_1} - \sigma_{r_2})\right] - \varepsilon_s + \rho\left[\varepsilon_0(T_1) - \varepsilon_0(T_2)\right] = 0 \tag{3.6}$$

By virtue of the fact that the failure takes place rapidly, this process may be regarded to be adiabatic, and, inasmuch as it is irreversible, it must be accompanied by an increase in entropy. In the model under consideration the entropy S per unit volume is

$$S = \int \frac{1}{T}\frac{d\varepsilon_0}{dT}\,dT + \int \frac{1}{T}\frac{d\varepsilon_s}{d\Sigma}\,d\Sigma \tag{3.7}$$

The temperature changes during failure are ordinarily small; it can be considered, therefore, that $\rho\varepsilon_0(T) = CT +$ const and that the surface energy does not depend on temperature. Then (3.7) becomes

$$S = C\ln T + \frac{\varepsilon_s}{T} + \text{const} \tag{3.8}$$

where C is the specific heat per unit volume of rock.

The change of entropy during failure is determined by Eq.

$$S_2 - S_1 = C\ln\frac{T_2}{T_1} + \frac{\varepsilon_s}{T_2} \tag{3.9}$$

(in the intact state $\varepsilon_s = 0$).

By the second law of thermodynamics

$$S_2 - S_1 > 0 \tag{3.10}$$

Using the relations (2.44) and (2.45), Eq. (3.6) can be reduced to the form

$$\varepsilon_s + C(T_2 - T_1) = \frac{\sigma_*^2}{2\gamma^2(3 - 4\gamma^2)\rho c_1^2}\left[1 + \frac{(1 - 2\gamma^2)/(1 - \gamma^2)}{(c_1/c_*)^2 - 1}\right] \equiv A, \quad c_* \equiv dr_1/dt \tag{3.11}$$

Determining T_1/T_2 from this result and substituting into (3.9), we obtain (3.10) in the form

$$\frac{\varepsilon_s}{CT_2} - \ln\left(1 + \frac{\varepsilon_s}{CT_2} - \frac{A}{CT_2}\right) > 0 \tag{3.12}$$

From its physical meaning ε_s is a positive quantity. The left-hand side of (3.12), considered as a function of ε_s/CT_2 for positive values, has a minimum at $\varepsilon_s/CT_2 = A/CT_2$ which is equal to A/CT_2. Then the condition (3.12) will be satisfied for all possible $\varepsilon_s \geqslant 0$ if the inequality $A > 0$ is satisfied, i.e. for

$$1 + \frac{(1-2\gamma^2)/(1-\gamma^2)}{(c_1/c_*)^2 - 1} > 0 \tag{3.13}$$

As will be shown below, the quantity ε_s is extraordinarily small compared to A. Accordingly, the condition (3.12) is equivalent to (3.13). The latter reduces to the inequalities:

$$|c_*| < c_1, \qquad c_0 = c_1\left(\frac{\gamma^2(3-4\gamma^2)}{1-\gamma^2}\right)^{1/2} < |c_*| < \infty \tag{3.14}$$

If, however, the relations $c_1 < |c_*| < c_0$ are satisfied, the inequality (3.13) (and then also (3.12)) will be violated; i.e. the solution will be thermodynamically inconsistent.

The thermodynamic limitations on the velocity of propagation of the failure front which have been obtained agree exactly with the limitations established above from purely mathematical considerations based on the requirement of uniqueness of evolution of the solution. This agreement is striking, although an analogous situation is well known in a simpler case, that of gas dynamics [13].

Let us now estimate the order of magnitude ε_s. As the failure front passes at a given distance $r = r_1$, let the radial cracks which occur split the rock into blocks with average dimension l. The surface area of a block normal to the radial direction will equal kl^2, where $k \sim 1$, and the number of blocks N will be determined by the relation $N = 4\pi r_1^2/kl^2$. The perimeter of a block equals ξl, where $\xi \sim 4$, so that in the volume $4\pi r_1^2 \Delta r_1$, the total area of lateral surfaces of all the blocks equals $N\xi l \Delta r_1$, and the total surface energy expended in forming this surface equals $\alpha N \xi l \Delta r_1$, where α is the unit surface energy for the rock. The surface energy of the rocks per unit volume equals $\alpha N \xi l \Delta r_1 : 4\pi r_1^2 \Delta r_1$, i.e., is

$$\varepsilon_s = \frac{\xi\alpha}{kl} \sim 4\,\frac{\alpha}{l} \tag{3.15}$$

If it is assumed that temperature changes do not occur during failure, i.e. that all the mechanical energy which is lost at the failure front is expended in the formation of the new surface, we obtain the estimate for l

$$l \sim 4\,\frac{\alpha}{A} \sim \frac{\alpha\rho c_0^2}{\sigma_*^2} \tag{3.16}$$

The magnitude of α for rocks, gypsum, calcite, corundum, etc., varies from 40 to 1500 erg/cm^2 [14], i.e. from $4\cdot10^{-5}$ to $1.5\cdot10^{-3}$ kg/cm. The magnitude of ρc_0^2 is of the order of 10^5 kg/cm^2, and the order of magnitude of σ_* varies from several tens to several thousands of kg/cm^2 [15]. Therefore, the maximum value of l is of the order of

$$l_{\max} \sim \frac{\max \alpha\rho c_0^2}{(\min \sigma_*)^2} \sim \frac{10^{-3}\cdot 10^5}{10^2}\ \text{см} = 1\ \text{см}$$

If, however, the quantities substituted into (3.16) are not the extreme values, but those corresponding to a given material, then considerably smaller values are obtained. Thus, for example, taking glass, we have $\alpha \sim 130$ erg/cm^2 $= 1.3\cdot10^{-4}$ kg/cm according to [14], and $\rho c_0^2 \sim 8\cdot10^5$ kg/cm^2, $\sigma_* \sim 250$ kg/cm^2 according to [15] so that $l \sim 1.6\cdot10^{-3}$ cm.

This shows that a large amount of cracking would be required for absorption of all the energy A, so that the dimensions of the blocks into which the rock is split turns out to be extremely small. Actually the dimensions of the blocks are many orders of magnitude larger than what is obtained under the assumption made above (these dimensions are no smaller than the dimensions of the natural inhomogeneities of the rock), so that the part ε_s of the energy which goes toward formation of the surfaces will be a negligibly small part of the energy loss A. The energy loss is almost entirely converted into heat (see (3.11)). The results which have been obtained are in complete agreement with the data of the paper [14], in which it was established by direct measurement that the energy expended in formation of new surface in the failure of solids is insignificantly small compared with the work done in causing failure, which is basically converted into heat. A case in which the fracture strength σ_* is anomalously small may constitute an exception. Even this case is improbable, since the quantity is also very small for weak materials. For crystals the strength is related to the surface energy by Polanyi's formula [16]

$$\alpha = \frac{\sigma_*^2}{2E} a \tag{3.17}$$

where E is Young's modulus and a is the interatomic distance. It is curious to note that this formula agrees with (3.16) up to a numerical factor if a is replaced by l. For crystals, σ_* is significantly larger than for rocks. Therefore l for rock is obtained larger by (3.16) than the interatomic distance occurring in (3.17).

We shall make one more remark of a general nature. The scheme for the solution of the problem of brittle fracture of rock developed here does not permit the determination of the dimensions of the blocks into which the rock is split by the cracks. As shown above, the role of surface energy in the failure process is insignificant. Therefore the dimensions of the blocks must be determined by using other mechanisms. At the present time, no rational proposals in this direction are known. However, there do exist some empirically established facts which should be considered in solving the problem. The experiments of Kuznetsov in the investigation of surface energy [14] have revealed the following fact. If fracture of a crystal of some material takes place under the action of a specified loading, the characteristic dimension of the pieces into which the crystal splits is proportional to the square root of the area to which the pressure causing fracture is applied. If it is assumed that this circumstance also holds true for the present problem of the fracture of rock, then, since the failure stress at the front $r = r_1(t)$ will depend only on $r_1/r_0 = x_1(\tau)$, the average dimension of the blocks l at a distance r_1 will be proportional to the square root of the area of the failure front, i.e.

$$l = \sqrt{4\pi r_1^2} K\left(\frac{\sigma_{r_1}}{\sigma_*}, \frac{\sigma_{r_2}}{\sigma_*}, \frac{\rho c_0^2}{\sigma_*}, \gamma^2\right) = r_0 K_1(x_1, \ldots) \tag{3.18}$$

where the form of the dependence of the coefficient K_1 on x_1 is determined by the dimensionless combinations found from the parameters characterizing the medium and the charge.

On the other hand, in experiments with loess soils [3], it was observed that under geometrically similar conditions and for the same explosive the dimensions l of the blocks were actually proportional to the scale of the phenomenon, i.e. to the radius of the charge r_0. This means that the property of the failure of crystals given above also holds for the failure of sizable volumes of rock under explosive loading. However, these empirical rules do not explain the physical nature of the phenomenon. Therefore, the question of the theoretical determination of the dimensions of the blocks during fracture of a solid remains unanswered. Attempts at solution of this problem must, of course, take into account the fact of the existence of the geometric similarity in fracture expressed by the relation (3.18). We remark in this regard that the relation (3.16) does not satisfy this geometric similarity. Therefore, the presence of the similarity in experiments indicates once more that the role of the expenditure of energy in the formation of new surfaces during fracture is an unimportant one.

4. We shall now turn to the analysis of the mathematical problem to which the original problem has been reduced. That it, we shall analyze the Eqs. (2.36), (2.37), (2.39) and (2.40). In this system one of the unknown functions $x_1(\tau)$ occurs in the arguments of the others, and this complicates the solution procedure. If, however, the solution of the system is known in a small vicinity of the point $\tau = \tau_2$, $x_1 = 1$, where τ_2 is the time when the failure front starts to move outward from the surface of the cavity, then the problem of construction of the solution of the system reduces to a sequence of initial value problems for certain systems of ordinary differential equations.

The solution in the vicinity of the point $\tau = \tau_2$, $x_1 = 1$ can be constructed by expanding all the unknown functions in series and substituting them into the system, which is then reduced to a set of algebraic equations for the determination of the coefficients of the series expansions. A number of the coefficients of the series expansions can be found directly from the system without actually substituting the series. Indeed, for $\tau = \tau_2$, $x_2 = 1$ we have

$$\xi = \xi_0 = \tau_2 - 1, \qquad \xi_1 = \xi_1^\circ = \xi_{10} = q\tau_2 - 1, \qquad \xi_2 = \xi_2^0 = \xi_{20} = q\tau_2 + 1$$

By virtue of the requirement of continuity of the displacements on the characteristic $\xi = \xi_0$, the values of $f(\xi)$ and $f'(\xi)$ must be continuous on it; i.e., the values of $f(\xi_0)$ and $f'(\xi_0)$ may be considered as known. We then find $f''(\xi_0)$ from Eq. (2.36) for $\xi = \xi_0$ and the value of $f_1(\xi_{10}) + f_2(\xi_{20})$ from (2.37).

In the equations for the displacements and stresses, only the sum $f_1(\xi_1) + f_2(\xi_2)$ occurs. Therefore, one of the quantities $f_1(\xi_{10})$ or $f_2(\xi_{20})$ can be specified arbitrarily (it can, for instance, be set equal to zero). After that the difference $f_1'(\xi_{10}) - f_2'(\xi_{20})$ can be found from (2.40).

We now note that at the initial time of formation of the failure front, the velocity of the front is zero. As a matter of fact, the stress σ_θ remains continuous across the characteristic $\xi = \tau - x = \tau_2 - 1$. On this characteristic not only $f(\xi)$ and $f'(\xi)$, but also $f''(\xi)$ will remain continuous. This means that σ_r will also be continuous. But since the value of $-\sigma_r$ on this characteristic for $x = 1$ is equal to the pressure in the cavity which changes continuously for $\tau = \tau_2$, the stress σ_r on the failure front must be continuous at the initial time. It is clear from Eq. (2.44) that the value of $\partial u/\partial r$ must then experience a discontinuity at the failure front, but it follows from Eq. (2.42) that the initial velocity of the failure front will be zero. The case in which $\sigma_* = 0$ is an exception requiring special consideration. This case was examined in the paper [9] under the assumption that there is no discontinuity of the functions at the failure front.

Thus in the relations (2.39), we have $x_1^{\cdot} = 0$ for $\tau = \tau_2$. If now Eq. (2.37) is differentiated with respect to τ and τ is set equal to τ_2, a relation is obtained which contains, in addition to known quantities, $f_1'(\xi_{10})$ and $f_2'(\xi_{20})$. Since the difference of these last quantities is known, they themselves can be found. As a result, all the quantities $f''(\xi_0)$, $f_1(\xi_{10})$, $f_1'(\xi_{10})$, $f_2(\xi_{20})$, $f_2'(\xi_{20})$ and $x_1^{\cdot}(\tau_2)$ are now known. This information is insufficient for the construction of the asymptotic solution. In addition, it is necessary to know at least the quantity $x_1^{\cdot\cdot}(\tau_2)$. To find this quantity it would be necessary to differentiate Eqs. (2.36), (2.37), (2.39) and (2.40) and set $\tau = \tau_2$. The four equations obtained in this fashion will contain the unknowns $x_1^{\cdot\cdot}(\tau_2)$, $f_1''(\xi_{10})$, $f_2''(\xi_{20})$ and $f'''(\xi_0)$ together with known quantities. These equations would then allow determination of the unknowns. However, this procedure proves to be impossible to carry out, for the quantities $x_1^{\cdot\cdot}(\tau_1)$, $f_1''(\xi_{10})$, and $f_2''(\xi_{20})$ are infinite. Therefore, the construction of the asymptotic solution requires the determination of the character of the singularities in the unknown functions at the point $x = 1$, $\tau = \tau_2$. Investigation shows that these singularities are of the form

$$x_1^{\cdot\cdot} \sim (\tau - \tau_2)^{-1/2}, \quad f_1'' \sim (\xi_1 - \xi_{10})^{-1/2}$$

$$f_2'' \sim (\xi_2 - \xi_{20})^{-1/2}, \quad f^{\mathrm{IV}} \sim (\xi - \xi_0)^{-1/2} \tag{4.1}$$

The coefficients of these asymptotic expressions may be easily determined.

We remark that in the vicinity of the point $\tau = \tau_2$, $x = 1$, the asymptotic solution cannot be determined from the equation of the failure front defined by the condition $\sigma_\theta(x_1, \tau) = \sigma_*$, where the left-hand side is given by Eqs. (2.6) and (2.10) because from these equations we can obtain the inequality

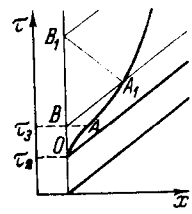
$$\left[\left(\frac{\partial\sigma_\theta}{\partial x}\right)_{\xi=\text{const}}\right]_{\substack{x=1\\ \tau=\tau_2}} = -\frac{1}{3-4\gamma^2}[(1-2\gamma^2)P + 4(1-\gamma^2)(\Sigma_* + P_0) +$$

$$+ 2\gamma^2(3-4\gamma^2)f(\xi_0)] < -\frac{1}{3-4\gamma^2}[(1-2\gamma^2)P + 4(1-\gamma^2)(\Sigma_* + P_0)] < 0$$

and then if the relation $x_1 = x_1(\tau)$ is determined in this way, we have $x_1^{\cdot}(\tau_2) < 1$; i.e. this relation cannot be the law of motion of the failure front. An exception is the case in which $\gamma^2 > \frac{1}{2}$ and the values of P, Σ_* and P_0 are such that the above inequality does not hold. In these cases the asymptotic solution should be constructed using the relation $x_1 = x_1(\tau)$ obtained in the manner just described.

The procedure for continuing the solution will be explained with the aid of a graphical scheme (Fig. 2).

Let the solution in the small triangle OAB be constructed by the method described earlier. Here OA is the initial part of the curve $x = x_1(\tau)$ and $\tau_3 - \tau_2$ is a very small quan-



Fig, 2

tity. We draw the characteristic $\xi_1 = q\tau - x = q\tau_3 - 1 = \xi_{11}$ from the point B. It intersects the curve $x = x_1(\tau)$ at some point A_1. Since the solution in the triangle OAB is known, the function $f_1(\xi_1)$ is known in the interval $\xi_{10} \leqslant \xi_1 \leqslant \xi_{11}$. The function $f_1(\xi_1)$ is then known in the Eqs. (2.36), (2.37) and (2.39) which have only $f(\xi)$, $f_2(\xi_2)$ and $x_1(\tau)$ as unknowns. Considering these three as functions of any one of the variables ξ, ξ_1, ξ_2, τ (say, ξ), and transforming the derivatives with respect to ξ, ξ_1, ξ_2 and τ in the equations to derivatives with respect to the variable chosen (ξ, in this case) using the transformation rules of the form $d.../d\xi_1 = (d.../d\xi)(d\xi_1/d\xi)^{-1}$, etc., we reduce these equations to a system of ordinary differential equations, for the determination of the four functions $f(\xi)$ $f_2(\xi)$, $x_1(\xi)$, $\tau(\xi)$.

A complete set of initial data is available from the conditions at the point A, so that the problem reduces to an initial value problem for the system of ordinary differential equations. Having solved this problem numerically, we find the segment AA_1 of the line $x = x_1(\tau)$ and the function $f(\xi)$ in the interval $\xi_0 \leqslant \xi \leqslant \xi_{A_1} = \tau_{A_1} - x_1(\tau_{A_1})$. The function $f_2(\xi_2)$ is then known in the interval

$$\xi_{20} \leqslant \xi_2 \leqslant \xi_{2A_1} = q\tau_{A_1} + x_1(\tau_{A_1})$$

We now draw the characteristic $\xi_2 = \xi_{2A_1}$ through the point A_1. It intersects the line $x = 1$ at the point B_1. Then in Eq. (2.40) the function $f_2(\xi_2^0)$ is known in the interval BB_1, so that the equation reduces to an ordinary differential equation (with independent variable $\xi_2^0 = q\tau - 1$) for the determination of the single function f_1. The initial condition is determined at point B. Solving this problem, we find the function f_1 for the new interval $\xi_{1B} \leqslant \xi_1 \leqslant \xi_{1B_1}$, after which we draw through the point B_1 the characteristic $\xi_1 = \xi_{1B_1}$, which intersects the front $x = x_1(\tau)$ further out at the point A_2. The problem of determination of the function $f(\xi)$, $f_2(\xi_2)$ and $x_1(\tau)$ for the segment A_1A_2 is the same as the preceding one; i.e. all the functions are determined from the system of equations already described but with new initial conditions given at the point A_1 and with a new interval of variation of the independent variable adjacent to the previous interval. After solving this problem, we go back to the line $x = 1$, etc.

By continuing this process it is possible to advance the solution of the problem as far as desired. However, the following must be kept in mind in the process of solution. The crack front continually radiates elastic waves. Therefore, the intensity of the discontinuity at this front and the front velocity decrease with time. Therefore, at some time the intensity and velocity of the front go to zero. The continuation of the solution beyond that time is carried out differently. It can be seen that at this time $x_1^{\cdot} = 0$ and by continuity $x_1^{\cdot\cdot} < 0$ so that if the procedure were continued, the front would begin to move backwards relative to the particles, "restoring" the failed material, which destroys the physical meaning of the solution. Therefore, at the time under consideration, it is necessary to set $x_1^{\cdot} = 0$ and to drop Eq. (2.36).

The solution obtained in this way corresponds to the fact that no new failure of material occurs and that the stress σ_θ at the boundary of the failed region toward the intact material falls off from the value σ_*.

Later either this stress remains positive and the solution is to be extended in the manner described to $\tau = \infty$ (asymptotic formulas are easily obtained for large values of τ), or there comes a time when this stress goes to zero. After this time, the condition $x_1^{\cdot} = 0$ must be dropped and replaced by Eq. (2.36), setting $\sigma_* = 0$ therein. The solution obtained corresponds to the fact that as a result of the change in sign of σ_θ on the boundary of the failed region, this boundary begins to move inside the failed material closing up the cracks, so that a compressive stress σ_θ appears in the material. In what follows, the boundary $x = x_1(\tau)$ generally oscillates about the position to which it tends as $\tau \to \infty$. If during this oscillation it goes beyond the true boundary between the intact and failed material, new fractures are possible and the procedure for the construction of the solution with cracking must be continued.

Finally, it is also necessary to verify whether the solution in the failed region satisfies the condition $-2\tau_* < \sigma_r < \sigma_*$. If at any time one of these conditions becomes an equality somewhere (the left one is more likely to do so), then from this point on it is necessary to construct a new failure front at which the material split-up by radial cracks fails again, by crushing. Generally speaking, this possibility is also unlikely to happen since all the stresses in the solution fall off with time. Therefore, if failure by means of shear (crushing) does not occur at the start, it is doubtful whether it can occur later. Nevertheless, verification is necessary because the solution of the problem can only be constructed numerically.

Let us now explain how the solution is to be constructed if the condition $\sigma_\theta = \sigma_*$ is reached at some point in the region $x > x_1(\tau)$. If on the curve determined by the relation $\sigma_\theta(x_*, \tau) = \sigma_*$, there is a part on which $|x_*^{\cdot}| > c_0$ and the condition (2.47) is satisfied, then above this part the functions f_1 and f_2 are found from Eqs. (2.37) and (2.38) in which $f(\xi)$ and $x_1(\tau) = x_*$ are known. The continuation of the solution for the succeeding intervals of variation of the arguments of the unknown functions corresponding to crossing the ends of the line $x = x_*(\tau)$ (as a result of either violation of Eq. (2.47) or the attainment of the condition $|x_*^{\cdot}| = c_0$) proceeds in a manner analogous to that described above. The only difference consists in the fact that a line $x = x_1 = x_*(\tau)$ generated at an interior point of failed material has two branches (two fronts, one propagating away from the cavity, one toward it). For the branch on which $x_*^{\cdot} > 0$ the procedure for solution is the same as that described above, except that the determination of the function f_1 is not carried out from a condition at $x = 1$, but from the corresponding condition on the branch $x_*^{\cdot} < 0$. However, this second branch causes a difference from what was given above, because on it the front $x = x_*(\tau)$ reflects back toward the cavity the elastic wave which moves out from the cavity into the intact material. Therefore, in the formulas for the solution of the problem for the intact material, a function corresponding to this reflected wave must be added. The formula for the displacements takes the form

$$U(x, \tau) = \frac{f'(\tau - x)}{x} + \frac{f(\tau - x)}{x^2} - \frac{F'(\tau + x)}{x} + \frac{F(\tau + x)}{x^2} \qquad (4.2)$$

The reflected wave either interacts with the cavity, if there is no failure in the vicinity of the cavity, or else with a failure front moving out from the cavity. The problem is thus made considerably more unwieldy, since it now requires construction of three failure fronts and five functions describing the motion in the failed and intact materials. In principle, however, the problem is solved as was described above for the simpler case. A special complication arises because of the fact that the failure front diverging from the cavity and the one converging toward it cause the interval in which the successive initial value problems are solved to become shorter and shorter as the fronts get closer and closer together. The intervals go to zero at the point where the fronts meet. This requires the development of a special method of calculation for the vicinity of this point.

We refer to the system of interactions in Fig. 3 for clarification of what has been stated Here AB is the failure front moving outward from the cavity; CB and DE are the converging and diverging fronts which develop at an interior point and which correspond to the phase when $|x_*^{\cdot}| < c_1$; CD is the part of these fronts corresponding to the phase $|x_*^{\cdot}| > c_0$; the various straight lines are characteristics; and the point B corresponds to the meeting of the two failure fronts.

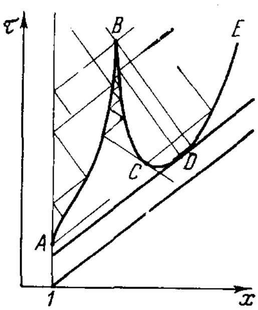


Fig. 3

Let us now pass on to the examination of the case in which the initial pressure is such that the point F falls on the line AB. Then, if p_{00} satisfies the inequality (2.12) and if, moreover, $p_{00} > p_{00F}$, then at some time τ_2 the difference $-p_{00} - \sigma\theta_1$ on the cavity becomes equal to $-2\tau_*$ and failure by means of shear begins. A failure surface $x = x_2(\tau)$ begins moving from the surface of the cavity into the material. On the side of this front toward the intact material the condition $\sigma_r - \sigma_\theta = -2\tau_*$ is satisfied. If, however, $p_{00} > p_0 + \gamma^{-2}\tau_*$ the failure begins at the initial time $\tau = 0$. Any further consideration must be carried out separately for the case when the strength of the rock is quite large and the case when it is quite small compared to $p_{00} - p_0$. In the first case the final radius of the failure zone is not very large compared to the radius r_0 of the charge. In the second case the radius is considerable. In the first case, the expansion of the cavity is also very slight and it is possible to continue the examination of the entire problem in Lagrangian coordinates since the differential equations which describe the motion of the medium remain linear in view of the smallness of the displacements and strains. In the second case, because of the considerable displacements in the vicinity of the cavity, the investigation in Lagrangian coordinates becomes inconvenient, for the linearity is lost and with it the possibility of reduction of the problem to ordinary differential equations.

We shall begin by examining the first case. On the side of the failure front toward the region of intact material, the condition $\sigma_r - \sigma_\theta = -2\tau_*$ is attained. Behind the front the material either undergoes plastic flow or elastic unloading as regards shear strains. Some yield condition is satisfied for plastic flow. The simplest one in the sense of making an effective solution possible is a yield condition which for the case of radial symmetry reduces to

$$\sigma_r - \sigma_\theta = -2\tau_{*1} \tag{4.3}$$

It is natural to suppose that the shear strength characteristic for failure by slip is not smaller than the magnitude τ_{*1} which characterizes the friction at the surface formed by the shear crack failure; that is, it is natural to consider that the following inequality is satisfied:

$$\tau_* \geqslant \tau_{*1} \tag{4.4}$$

The failed material which is in a state of plastic flow will be described by the yield condition (4.3) and by Hooke's law for the volumetric strain

$$\frac{1}{3}(\sigma_r + 2\sigma_\theta) = \left(\lambda + \frac{2}{3}\mu\right)\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) - p_0 \tag{4.5}$$

Substitution of (4.3) and (4.5) into the equation of motion (2.2) taking account of (2.3), leads to Eq.

$$\frac{\partial^2 U}{\partial \tau^2} = \left(1 - \frac{4}{3}\gamma^2\right)\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x} + \frac{2U}{x}\right) - \frac{4T_{*1}}{x}, \quad \tau_{*1} = \rho c_0^2 T_{*1} \tag{4.6}$$

The general solution of this equation has the form

$$U = \frac{F_1'(q_1\tau - x) - F_2'(q_1\tau + x)}{x} + \frac{F_1(q_1\tau - x) + F_2(q_1\tau + x)}{x^2} + \frac{4T_{*1}}{3(1 - {}^4/_3\gamma^2)}x\ln x, \qquad q_1 = \sqrt{1 - {}^4/_3\gamma^2} \tag{4.7}$$

where F_1 and F_2 are arbitrary functions of their arguments. The stresses are found from Formulas

$$\sigma_r = -\rho c_0^2\left[\left(1 - \frac{4}{3}\gamma^2\right)\frac{F_1'' + F_2''}{x} - 4T_{*1}\ln x + P_0\right], \quad \sigma_\theta = \sigma_r + 2\tau_{*1} \tag{4.8}$$

The functions F_1 and F_2 must be found from the boundary conditions at the failure front and either the condition at the cavity if slip occurs right up to the cavity, or the condition at an unloading front which separates the region occupied by the failed material in which slip occurs plastically and a part in which elastic unloading in shear takes place. Such an unloading front may, in general, occur.

Let us consider the conditions at the failure front. We have for the stresses ahead of and behind the front, respectively,

$$\begin{aligned} \sigma_{r_1} &= \rho c_0^2\left(1 - \frac{4}{3}\gamma^2\right)\left(\frac{\partial u_1}{\partial r} + \frac{2u_1}{r}\right) - \frac{4}{3}\tau_* - p_0 \\ \sigma_{r_2} &= \rho c_0^2\left(1 - \frac{4}{3}\gamma^2\right)\left(\frac{\partial u_2}{\partial r} + \frac{2u_2}{r}\right) - \frac{4}{3}\tau_{*1} - p_0 \end{aligned} \tag{4.9}$$

Using the condition of continuity of the displacements and the theorem of impulse-momentum, we obtain with the aid of (4.9)

$$\frac{\Delta\rho_2 - \Delta\rho_1}{\rho} = \frac{\partial u_1}{\partial r} - \frac{\partial u_2}{\partial r} = \frac{\sigma_{r_1} - \sigma_{r_2}}{\rho c_*^2} = \frac{4}{3}(\tau_* - \tau_{*1})(\rho c_*^2)^{-1}\left[\left(1 - \frac{4}{3}\gamma^2\right)\left(\frac{c_0}{c_*}\right)^2 - 1\right]^{-1} \tag{4.10}$$

Considering (4.4), we conclude from (4.10) that for $|c_*| < c_0 (1 - 4/3\gamma^2)^{1/2} \equiv c_2$, we have $\Delta\rho_2 > \Delta\rho_1$; i.e., the failure front will be a condensation front (will involve a positive jump in density). For $|c_*| > c_2$ it will be a rarefaction front. It appears natural from physical considerations to assume that the second case is impossible, for the friction forces decrease during failure ($\tau_* > \tau_{*1}$) and the possibility of additional compression arises. If this limitation on the velocity $|c_*|$ is not made, then if the limiting state $\sigma_{r_1} - \sigma_{\theta_1} = -2\tau_*$ occurs at an interior point $x > 1$ the construction of the solution in the vicinity of this point becomes non-unique. Indeed, the limiting state first occurs at a point where $|c_*| = \infty$ (the minimum point of the line $\tau = \tau(x)$ defined by the relation $\sigma_{r_1}(\tau, x) - \sigma_{\theta_1}(\tau, x) = -2\tau_*$) and it can be assumed that in the vicinity of this point this line is an actual failure front up to the points where $|c_*| = c_0$, or else it is necessary to construct two branches of a failure front from this point satisfying the condition $|c_*| < c_2$ (see the analogous condition for the case of failure by the formation of tension cracks). The assumption made above prescribes the choice of the second of these possibilities. If the first one is chosen, then by virtue of the fact that

$$\sigma_{\theta_1} - \sigma_{\theta_2} = \frac{2(\tau_* - \tau_{*1})}{(c_2/c_*)^2 - 1}\left[\left(\frac{c_2}{c_*}\right)^2 - \frac{1}{3}\right]$$

we have $\sigma_{\theta_1} < \sigma_{\theta_2}$ and $\sigma_{r_1} < \sigma_{r_2}$ for $c_2 < |c_*| < \sqrt{3}\, c_2$; i.e., the two stresses as well as the density decrease during failure, which seems improbable.

Let us again examine the energy equation. Proceeding as in the case of failure by tension cracking, we obtain

$$C(T_2 - T_1) = \frac{2(\tau_* - \tau_{*1})}{3\gamma^2\rho c_0^2}\left\{1 + \frac{4/3\,\gamma^2}{1 - 4/3\,\gamma^2}\left[\left[(1 - 4/3\,\gamma^2)\left(\frac{c_0}{c_*}\right)^2 - 1\right]^{-1} + 1\right]\right\} \quad (4.11)$$

The condition of increase of entropy, i.e., the condition $T_2 - T_1 > 0$, leads, by virtue of (4.4) to the inequality

$$|c_*| < c_0\sqrt{1 - 4/3\,\gamma^2} = c_2, \qquad |c_*| > c_0 \quad (4.12)$$

That is, it once more leads in a striking manner to the inequalities obtained from the mathematical requirement that the solution of the problem unfold uniquely.

It seems that this is related to the fact that the requirement of unique evolution of the solution of the equations of continuum mechanics can apparently be formulated as a condition of positive change with time of some quadratic functional related in a certain way to the form of the equations, so that this functional is a monotonously increasing function of the entropy of the system which is introduced with the aid of independent thermodynamic considerations. It would be interesting to examine these questions in their general mathematical formulation. The solution of this problem would allow introduction of the concept of entropy by purely mathematical means for a given system of partial differential equations. The requirement of non-decrease of this entropy would then guarantee the unique solubility of problems for the partial differential equations.

Returning to the present problem, we remark that thermodynamic limitations do not exclude the possibility $|c_*| > c_2$, which leads to a rarefaction during failure. However, in the solution of problems, the limitation $|c_*| < c_2$ suggested above should apparently be retained.

We shall now assume that the shear is everywhere plastic behind the failure front. The condition for this is the positiveness of the expression Λ of [2], which reduces in the present case ($F(p) = \text{const}$) to the condition

$$\Lambda \equiv (\sigma_r - \sigma_\theta)\frac{\partial}{\partial\tau}\left(\frac{\partial U}{\partial x} - \frac{U}{x}\right) =$$

$$= 2\tau_{*1}\sqrt{1 - \frac{4}{3}\gamma^2}\left[\frac{F_1''' + F_2'''}{x} - 3\left(\frac{F_1'' - F_2''}{x^2} + \frac{F_1' + F_2'}{x^3}\right)\right] \geqslant 0 \quad (4.13)$$

In examination of the solution, the sign of Λ must be traced. It is necessary to construct a wave front behind which the shear once more becomes elastic starting with the point at which the relation $\Lambda = 0$ is first reached.

We note that at the time when the unloading front overtakes the failure front no subsequent failure will generally occur. Indeed, since the velocity of propagation of disturbances in the unloading region is the same as in the intact material (i.e. is equal to c_0), the failure front should become a characteristic after it is overtaken. That is, its velocity should become c_0, since if the velocity of small disturbances on both sides of a strong discontinuity is the same, the velocity of the discontinuity itself can only be equal to this velocity of disturbances. However, in this case, the quantity $\sigma_{r1} - \sigma_{\theta 1}$ along the characteristic generally decreases so that the condition $\sigma_{r1} - \sigma_{\theta 1} = -2\tau_*$ is not satisfied and the failure ceases. It follows from what has been said that if failure takes place, then the shear is necessarily plastic behind the front. An unloading front can arise either at the cavity or within the failed region and afterwards can overtake the failure front and suppress it. (It is true that a strong discontinuity will propagate along the characteristic which emerges from the point where the unloading wave overtakes the failure front).

By virtue of the fact that $|c_*| < c_2 < c_0$ a failure front arising at the surface of the cavity radiates an elastic wave into the intact material even in the case when the failure commences at the initial instant $\tau = 0$ ($p_{00} > p_0 + \gamma^{-2}\tau_*$). As time increases, either a wave of unloading arising behind the failure front overtakes it and stops the failure or this does not occur. However, because of the decrease of the stresses σ_{r1} and $\sigma_{\theta 1}$, a time must come in either case when the point $\sigma_{r1}, \sigma_{\theta 1}$ reaches the point B (Fig. 1) and at that time the failure front bifurcates; a front moves ahead on which failure by the formation of tension cracks occurs, and behind it a front propagates on which a second failure by crushing occurs.

The procedure for the construction of the solution up to the time of bifurcation is completely analogous to the one considered above for the case of failure by the formation of radial cracks. In order to continue the solution beyond the time of bifurcation of the front, the asymptotic behavior of the solution of the problem in the vicinity of the bifurcation point must be examined by means of series expansion about this point and determination of the type of possible singularities of the unknown functions. After this the procedure of reduction of the problem to a set of initial value problems for ordinary differential equations must again be applied. The suitable systems of functional-differential equations are easily written out by using the formulas given above. However, we shall not do this here. The construction of the solution for the late stages when the failure fronts decay can also be carried out by the method described above for the case of a single front of failure by formation of radial cracks. It is only necessary to remark that after the velocity of the crushing front goes to zero, no motion to restore it takes place and it must be considered a contact discontinuity after that.

If a wave of unloading occurs, the equations of motion behind it are determined by Hooke's law in the form

$$\sigma_r = \sigma_{r*} + \lambda\left(\frac{\partial u}{\partial r} - \frac{\partial u_*}{\partial r} + 2\,\frac{u - u_*}{r}\right) + 2\mu\left(\frac{\partial u}{\partial r} - \frac{\partial u_*}{\partial r}\right)$$
$$\sigma_\theta = \sigma_{\theta*} + \lambda\left(\frac{\partial u}{\partial r} - \frac{\partial u_*}{\partial r} + 2\,\frac{u - u_*}{r}\right) + 2\mu\,\frac{u - u_*}{r} \qquad (4.14)$$

where σ_{r*}, $\sigma_{\theta*}$, $\partial u_*/\partial r$, and u_*/r are the values of the stresses and strains on the unloading front, which depend only on x.

Substitution of (4.14) into the equation of motion (2.2) reduces the latter to the form

$$\frac{\partial^2 U}{\partial \tau^2} = \frac{\partial^2 U}{\partial x^2} + \frac{2}{x}\frac{\partial U}{\partial x} - \frac{2}{x^2}U + \varphi(x) \qquad (4.15)$$

where $\varphi(x)$ is expressed in terms of σ_{r*}, $\sigma_{\theta*}$, $\partial u_*/\partial r$, and u_*. It is easy to write out the general solution of this equation

$$U = \frac{\psi_1'(\tau - x) - \psi_2'(\tau + x)}{x} + \frac{\psi_1(\tau - x) + \psi_2(\tau + x)}{x^2} + U_0 \qquad (4.16)$$

where ψ_1 and ψ_2 are arbitrary functions and $U_0(x)$ is a solution of the nonhomogeneous Eq. (4.15), which is expressed in terms of $\varphi(x)$ by quadratures.

$$U_0(x) = \frac{1}{x^2}\int^{x} \zeta^2\left(\int^{\zeta} \varphi(\xi)\,d\xi\right)d\zeta \qquad (4.17)$$

In the solution of the boundary value problem, σ_{r*}, $\sigma_{\theta*}$, $\partial u_*/\partial r$, and u_* are expressed in terms of the solution in the region where plastic shear occurs and of the unknown law of motion of the unloading front $x = x_*(\tau)$. Therefore, in the formulas for the unloading region

ψ_1, ψ_2, and $x_*(\tau)$ are undetermined. There are three equations in these three functions: the unloading condition $\Lambda = 0$ and the two equations of continuity for $\partial u/\partial r$ and u which guarantee the continuity of stresses and strains for $x = x_*(\tau)$. The problem for ψ_1, ψ_2, and x_* which arises in this way is solved jointly with the problems for the failure fronts described above by the method of reduction to a set of initial value problems for ordinary differential equations.

For porous materials of low strength, the yield condition should be taken in the more general form

$$I_2 = [k(p - p_0) + b]^2 \tag{4.18}$$

In the case of central symmetry, this reduces to the relation

$$\sigma_\theta = \alpha\sigma_r + \beta \tag{4.19}$$

The case of rock with high strength considered above corresponds to $\alpha = 1$. The limiting value of α, which corresponds to an elastic material (Hooke's law) is equal to $\alpha_e = 1 - 2\gamma^2$. Therefore, the range of possible values of α is determined by the inequalities

$$1 - 2\gamma^2 \equiv \alpha_e < \alpha \leqslant 1 \tag{4.20}$$

Like the yield condition, the strength condition (failure condition) for the general case is written in the form

$$I_2 = \Phi(p - p_0) \tag{4.21}$$

which in particular can be reduced to the form (4.19) for porous rocks under moderate stresses. Experimental data on the strength of a number of rocks (limestones, shales, Carrara marble, and others) published in [17] confirm what has been stated and also the comment made in Section 1 on the decrease of the function $\Phi(p - p_0)$ as p increases, for hard porous rocks.

The procedure for construction of the solution for the case of weak rocks is a combination of the methods used in the present paper and in [11].

We note that for nonporous and not very strong materials the relation between hydrostatic pressure and volumetric strain is linear and reversible. Therefore, in the initial stage of the motion, as long as the velocity of the failure front is commensurate with the acoustic velocity and the motions are still small, the problem can be solved in Lagrangian coordinates. Only when this velocity becomes small so that the compressibility can be neglected [18] everywhere in the failed zone is it necessary to transform to Eulerian coordinates and to use the method of the paper [11]. For porous materials which have considerable irreversible compressibility, the method of [11] must be applied from the very start.

Finally, there is a difference from the problem of [11] in that the yield condition and the failure condition do not coincide. Therefore, the relation (4.21) describing the process of radiation of an elastic wave during failure is different in the present case from what was used to describe the motion behind the shock wave.

It should be remarked that in the case of nonporous materials when $\alpha \neq 1$ in the yield condition and it is necessary to use Lagrangian coordinates for the initial stage, the fundamental equation for U is, unfortunately, not integrable in the general form with two arbitrary functions. For this stage it is possible to set $\alpha = 1$ and to obtain the solution with some error, in the hope that the initial stage is of short duration and the error does not have time to accumulate.

We make a final comment in the following connection. In [4] it is pointed out that for comparatively small stresses, as experiment shows, rocks exhibit plasticity without failing, so that in the vicinity of the leading front of the disturbances the wave parameters damp out more strongly than follows from the solution of the problem under the assumption of elastic behavior of the material.

The asymptotic theory of [4] shows that plastic deformation is localized in a narrow layer adjoining the leading front of the disturbances. In this layer the parameters of the motion increase sharply up to a maximum and then change smoothly. This corresponds to the case where $\alpha \approx \alpha_e$ in the yield condition, so that the front of transition to the plastic state is located very close behind the lead characteristic.

In this case, the wave of unloading of shear deformation also follows closely behind this characteristic, so that the unloading condition $\partial v/\partial r - v/r = 0$ is satisfied near the point (behind it) where $\partial v/\partial r = 0$. Since the layer in which the parameters change sharply and arrive at a maximum (i.e. unloading starts in it) is very thin, it is possible to neglect this thin layer in constructing the solution in the much larger region back of it. That is, it is possible to consider that right up to the lead characteristic of the linear Eq. (4.15), the disturbances σ_{r*}, $\sigma_{\theta*}$ and $\partial u_*/\partial r$ are given as functions of x. These distributions may be taken either as those given in [4] by an asymptotic analysis, or from experiment. The functions ψ_2 and $U_0(x)$ in Eq. (4.16) are then known. In the rest of the construction, the formulas for the elastic wave in the intact material, which are needed in the analysis of the failure fronts and of the motion of the rock behind them, contain not only the direct wave radiated by the failure front, but also a wave reflected from the unloading wave, i.e., from the lead characteristic. This reflected wave corresponds to the function $\psi_2(\tau + x)$. It is possible to do all this, but it is not necessary, for the reality of the occurrence of plastic flow in the vicinity of the front in [4] has not been proved. It is only a hypothesis which may not correspond to the actual situation. It is not ruled out that the effects observed in the experiment mentioned in [4] can be completely explained within the framework of the scheme adopted in the present paper.

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